S. S. College. Jehanabad (Magadh University)

Department : Physics

Subject : Quantum Mechanics

Class : B.Sc(H) Physics Part III

Topic: The Linear Harmonic Oscillator

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The linear Harmonic oscillator

The Force acting on a particle executing linear harmonic oscillation is given by

$$F = -Kx \text{ and}$$

Potential energy corresponding to this force is

$$V = \tfrac{1}{2}Kx^2 \quad ; \omega = \sqrt{K/m} \; ; K = m\omega^2$$

$$V(x) = \tfrac{1}{2}m\omega^2 x^2$$

The time independent schrodinger equation for Harmonic oscillator

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \tfrac{1}{2}m\omega^2 x^2 \psi(x) = E\psi(x) \quad -(1)$$

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}\left(E - \tfrac{1}{2}m\omega^2x^2\right)\psi(x) = 0$$

$$\frac{d^2\psi(x)}{dx^2} + \frac{m\omega}{\hbar}\left[\frac{2E}{\hbar\omega} - \frac{m\omega}{\hbar}x^2\right]\psi(x) = 0 \quad -(2)$$

$$\lambda = \frac{2E}{\hbar\omega} \text{ and } \xi = \alpha x, \alpha = \left(\frac{m\omega}{\hbar}\right)^{1/2}$$

$$\frac{d\xi}{dx} = \alpha \; ; \; \frac{d\psi}{dx} = \frac{d\psi}{d\xi}\frac{d\xi}{dx} = \alpha\frac{d\psi}{d\xi}$$

$$\frac{d^2\psi}{dx^2} = \alpha\frac{d}{d\xi}\left(\frac{d\psi}{dx}\right) = \alpha^2\frac{d\psi}{d\xi} = \frac{m\omega}{\hbar}\frac{d^2\psi}{d\xi^2}$$

Putting these values in eq (2), we get

$$\frac{m\omega}{\hbar}\left[\frac{d^2\psi}{d\xi} + (\lambda - \xi^2)\psi\right] = 0 \quad -(3)$$

The solution of above equation (3) is given by Hermite equation

$$\psi(\xi) = e^{-\xi^2/2} H(\xi) \quad -(4)$$

$$\frac{d^2 H(\xi)}{d\xi^2} - 2\xi \frac{dH(\xi)}{d\xi} + (\lambda - 1) H(\xi) = 0 \quad -(5)$$

$$H(\xi) = \sum_{K=0}^{\infty} a_K \xi^K = a_0 + a_1 \xi + a_2 \xi^2 + \cdots$$

$$H(\xi) = (a_0 + a_2 \xi^2 + a_4 \xi^4 + \cdots) + (a_1 \xi + a_3 \xi^3 + \cdots) \quad -(6)$$

If $\xi \to \infty$, then higher term in series dominate.

$$\frac{a_{K+2}}{a_K} \to \frac{2}{K} \text{ for large } K$$

and

$$e^{\xi^2} = \sum_{K=0,2,4} b_K \xi^K \; ; \; b_K = \frac{1}{(K/2)!}$$

Then

$$\frac{b_{K+2}}{b_K} = \frac{(K/2)!}{(K+2/2)!} = \frac{2}{K+2} \to \frac{2}{K} \text{ for large } K$$

$$\boxed{\psi(\xi) = e^{\xi^2/2}} \text{ for large } K.$$

Thus for the acceptable solution; the series must be terminated to a polynomial and it can happen when λ is an odd integer:

$$\lambda = 2n+1$$

The energy Eigen value

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega \;, \quad n = 0, 1, 2 \cdots$$

At $n = 0$,

$E_0 = \frac{1}{2}\hbar\omega \to$ ground energy or zero point energy. It is characteristics of quantum mechanics.

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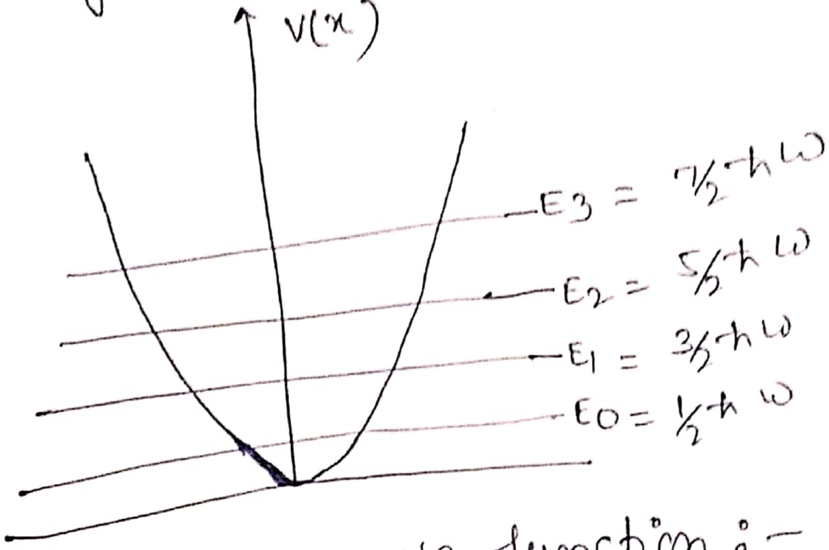
It may be noted that eigenvalues are non degenerate because for each value of quantum number, n there exists only one eigen-function.

The potential well and the five lowest energy eigenvalues of harmonic oscillator:



Harmonic oscillator wave function :-

$$\psi_n(\xi) = e^{-\xi^2/2} H_n(\xi)$$

$$\psi_n(x) = N_n e^{-\frac{\alpha^2 x^2}{2}} H_n(\alpha x)$$

$$\int_{-\infty}^{\infty} |\psi_n(x)|^2 dx = \frac{|N_n|^2}{\alpha} \int_{-\infty}^{\infty} e^{-\xi^2} H_n^2(\xi)\, d\xi = 1$$

$$N_n = \left(\frac{\alpha}{\sqrt{\pi}\, 2^n n!}\right)^{1/2}$$

$$\psi_n(x) = \left(\frac{\alpha}{\sqrt{\pi}\, 2^n n!}\right)^{1/2} e^{-\alpha^2 x^2/2} H_n(\alpha x)$$

$$\psi_0(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} \exp\left(-\tfrac{1}{2}\alpha^2 x^2\right)$$

$$\psi_1(x) = \left(\frac{2\alpha}{\sqrt{\pi}}\right)^{1/2} \alpha x \exp\left(-\tfrac{1}{2}\alpha^2 x^2\right)$$

$$\psi_2(x) = \left(\frac{\alpha}{3\sqrt{\pi}}\right)^{1/2} \left(2\alpha^3 x^3 - 3\alpha a x\right) \exp\left(-\tfrac{1}{2}\alpha^2 x^2\right)$$